Towards a Quantum de Finetti Theorem

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Overview of Presentation

- What is the dF Theorem and why do we need it?

- Uses of the theorem in quantum logic and quantum information theory

- Tackling the theorem in both the finite and infinite cases

The dF theorem answers a basic question in the Bayesian view of probability:

What is an *unknown* state?

Given a physical *system*, any observer has some belief about it, which is encapsulated in the *state*. The state cannot be unknown to the observer.

On the other hand, we're usually tempted to think of the state as a property of the system, a property which we can learn by making measurements to sort things out. Measurement reveals the state to us.

But if we are in charge of the state—it represents our beliefs about the system—what is it we're doing when we make such a measurement?

Example

Bond plays roulette in Monte Carlo.
Is the roulette wheel fixed?

To decide, watch the outcomes.
Compare with initial probability for unfair wheel.

Probability for the wheel to be fixed
$\Rightarrow$ *Probability* for the *probabilities* of outcomes.

Bond is using an unknown state. He hopes observation will indicate the “true” state.

In the Bayesian view, talk of probability of probabilities is absurd.

de Finetti theorem:

$$W_{\text{exch}} = \int dw\, p(w)\, w^{\otimes N}$$

unknown state $\leftrightarrow$ *exchangeable* sequence of measurements.

Exchangeable: order of measurements doesn't matter.

Exchangeability is an *assumption* about the global state W.

- Bond believes $\int \mathrm{d}w\, p(w)\, w^{\otimes N}$

- de Finetti: W is *exchangeable*

dF essential to Bayesian physical theory:

allows the convenient fiction of unknown state

Connection to Quantum Logic

- dF theorem is a "must have" feature

- characterize logical systems compatible with physical theory

- expose differences/similarities between classical and quantum

Other uses

- dF theorem succinctly characterizes permutation invariant states
  $\Rightarrow$ quantum cryptography

- possibly useful for characterizing entanglement

Tackling the theorem

Two cases are of independent interest

- Finite exchangeability
  $\Rightarrow$ *Affine* combinations of i.i.d. states

- Infinite exchangeability
  $\Rightarrow$ *Convex* combinations of i.i.d. states

Finite Exchangeability

- States are vectors in a space $\mathcal{H}_n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$)
  - classical: $\mathbb{R}^d$ for d outcomes
  - quantum: $\mathbb{R}^{d^2-1}$ for d levels
- Consider $\mathcal{S}_d^N$: symmetric subspace of $\mathcal{H}_d^N$
- Via Peng & Waldron (2002):

  $\{V_i = v_i^{\otimes N} \in \mathcal{S}_d^N\}$ is a basis for $\mathcal{S}_d^N$ for almost all choices of $\{v_i\}$

$\Rightarrow$ For $W \in \mathcal{S}_d^N$, W has a de Finetti form

$$W = \int \mathrm{d}w\, q(w)\, w^{\otimes N}$$

$\Rightarrow \int \mathrm{d}w\, q(w) = 1$ by normalization

Peng & Waldron Theorem

For almost every $v_1, v_2, \ldots, v_m \in \mathbb{R}^d$

$$\text{rank}(\langle v_i, v_j\rangle^r) = \min\{\binom{r+d-1}{d-1}, m\}$$

- $\langle v_i, v_j\rangle = G$ is the Gram matrix

  rank(G) = # of lin. indep. vectors

- $\langle v_i, v_j\rangle^r = G^{\circ r}$ is the rth Hadamard power

  $\Rightarrow$ # of indep vectors in set $\{v_1^{\otimes r}, \ldots v_m^{\otimes r}\}$

- $\dim(\mathcal{S}_d^N) = \binom{N+d-1}{d-1}$

$\Rightarrow$ Choose $m = \binom{N+d-1}{d-1}$, $r = N$

Infinite Exchangeability

- Heath & Sudderth proof for classical states

  Proof idea: unimaginably large urns are essentially i.i.d. states, e.g.,

  with replacement $\approx$ *without* replacement

- Outline of quantum Heath & Sudderth proof
  - Examine quantum urns
  - Investigate asymptotic relation to i.i.d. states.

Heath & Sudderth Proof
for binary variables

- Symmetrize the probability distribution:

$$p(x_1, x_2, \ldots, x_n) \to p(r|n)$$

- Consider the distribution as a marginal:

$$p(r|n) = \sum_{s=0}^{m} p(s|m) \binom{n}{r} \frac{(s)_r (m-s)_{n-r}}{(m)_n}$$

$$(s)_r = \tfrac{s!}{(s-r)!}$$

- Let $m \to \infty$

$$p(r|n) = \int \mathrm{d}z\, p_m(z) \binom{n}{r} \frac{(zm)_r [(1-z)m]_{n-r}}{(m)_n}$$

$$p_m(z) = \textstyle\sum_{s=0}^{m} p(zm|m) \delta(z - s/m)$$

- In the limit

$$p(r|n) = \int \mathrm{d}z\, p_\infty(z) \binom{n}{r} z^n (1-z)^{n-r}$$

Quantum H & S Proof Outline

First, symmetrize the state:

$$[U(\pi), \rho^{(N)}_{\mathsf{sym}}] = 0 \quad \text{for all permutations } \pi$$

$$\Rightarrow \quad \mathcal{H}_d^N = \bigoplus_\lambda V_u^\lambda \otimes W_p^\lambda$$

- Permutations and N-fold unitaries commute
- λ labels irrep of permutation group
- $U(\pi) \;\rightarrow\; \bigoplus_\lambda \mathbb{I}_u \otimes D(\pi)_p^\lambda$
- $U(g)^{\otimes N} \;\rightarrow\; \bigoplus_\lambda D(g)_u^\lambda \otimes \mathbb{I}_p^\lambda$

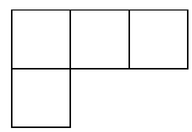
$$\Rightarrow \; \rho^{(N)}_{\mathsf{sym}} = \bigoplus_\lambda p_\lambda \int \mathsf{d}\psi \, \mathsf{p}(\psi) \, |\psi\rangle\langle\psi|_\lambda \otimes \mathbb{I}$$

Quantum H & S Proof Outline

Skip marginalization for now

Note result of Keyl & Werner (2001)

- recall λ is a Young diagram with d rows and N boxes



- let $P_\lambda : \mathcal{H}_d^N \to V_u^\lambda \otimes W_\pi^\lambda$

- $\lim_{N \to \infty} \mathrm{tr}[P_\lambda \, \rho^{\otimes N}] = 1$

  when $\lambda / N \to$ eigenvalues of ρ.

- Asymptotically $\rho_{\mathrm{sym}}^{(N)} \rightsquigarrow \rho^{\otimes N}$?