Measurement-Based Quantum Computation in Realistic Spin-1 Chains

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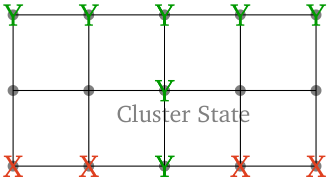
Measurement-Based Quantum Computation

Single-site mmts on an entangled state

instead of

unitary gates on an unentangled state

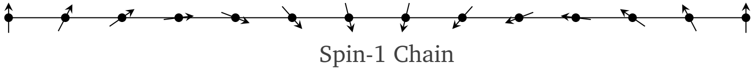


- ☞ New avenue in QC research — which states are useful?
- ☞ Practical implications — can we engineer these states?
- ☞ Connections to condensed matter — maybe ground states of realistic Hamiltonians are useful?

Explore the AKLT model

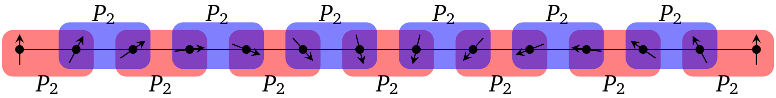
Outline:

1. The AKLT model and how to perform single-qubit logic
   Brief look at physical implementation
2. How to characterize computational ability of imperfect states
   Numerical results: Computational ability is robust!
3. Improving performance: gate buffering and AKLT "renormalization"
4. Summary and Outlook



Affleck-Kennedy-Lieb-Tasaki: Consider the nearest-neighbor Hamiltonian

$$H_{\mathrm{AKLT}} = \sum_{j=1}^{n} (\vec{S}_j \cdot \vec{S}_{j+1}) + \tfrac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2 \simeq \sum_{j=1}^{n} (P_2)_{j,j+1}$$



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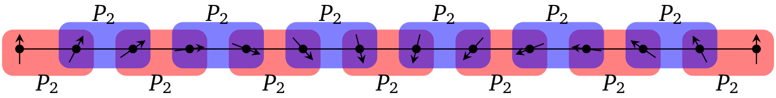

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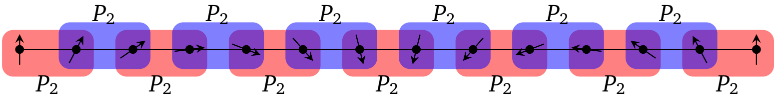
- Ground state is unique periodic BCs or $n \to \infty$; 4fold degenerate open BCs and $n < \infty$
- Gap to first excited state (conjectured by Haldane, analytic example by AKLT)
- Ground state is a “valence bond solid” (VBS), frustration-free



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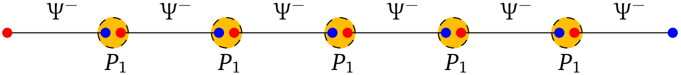

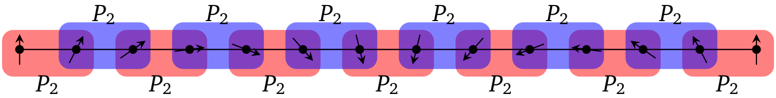

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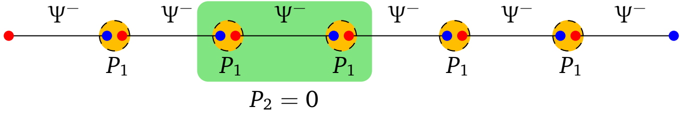
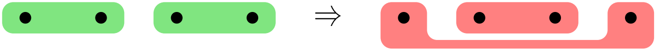
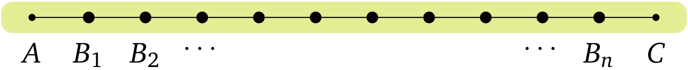


VBS is the original matrix-product state

- Easy to work out using entanglement swapping relations



- For $|s\rangle \equiv |m_s = 0\rangle$, $s = x, y, z$ &
  Pauli operators σ_s (but with $\sigma_y = \sigma_x \sigma_z$):



$$|\mathcal{G}_{\text{open}}\rangle = \sum_{\{s_k\}} |s_1, s_2, \ldots, s_n\rangle_B \otimes \left(\sigma_{s_n} \sigma_{s_{n-1}} \cdots \sigma_{s_1}\right)_C |\Psi^-\rangle_{AC}$$

$$|\mathcal{G}_{\text{periodic}}\rangle = \sum_{\{s_k\}} \text{Tr}\left[\sigma_{s_n} \sigma_{s_{n-1}} \cdots \sigma_{s_1}\right] |s_1, s_2, \ldots, s_n\rangle$$

Single-qubit logic on the VBS

$$|\mathcal{G}\rangle = \sum_{\{s_k\}} |s_1, s_2, \ldots, s_n\rangle_B \otimes \left(\sigma_{s_n}\sigma_{s_{n-1}} \cdots \sigma_{s_1}\right)_C |\Psi^-\rangle_{AC}$$

☞ Chain encodes one logical qubit (think of it at C)

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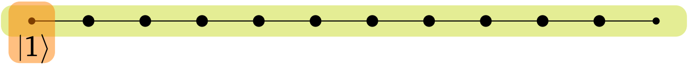

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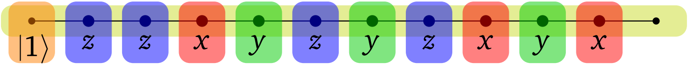

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- ☞ To do nothing, measure in the $|s\rangle$ basis
  “Byproduct” operators accumulate on C (change of basis)

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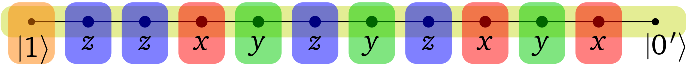

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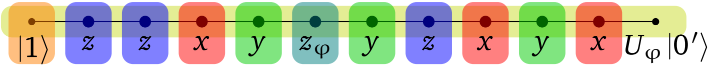

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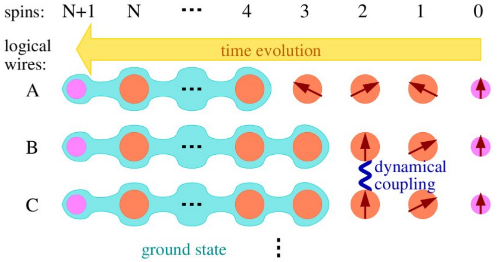
- ☞ Chain encodes one logical qubit (think of it at C)
- ☞ Initialize: Measure $|0\rangle$, $|1\rangle$ on end qubit A
- ☞ To do nothing, measure in the $|s\rangle$ basis
  "Byproduct" operators accumulate on C (change of basis)
- ☞ To rotate, measure in a rotated basis. Example: $R_{\hat{x}}(\varphi)$

  $\{|x_\varphi\rangle = |x\rangle\,, |y_\varphi\rangle = \cos\frac{\varphi}{2}\,|y\rangle + i\sin\frac{\varphi}{2}\,|z\rangle\,, |z_\varphi\rangle = i\sin\frac{\varphi}{2}\,|y\rangle + \cos\frac{\varphi}{2}\,|z\rangle\}$.
  Byproducts are σ_x, σ_y, and σ_z, respectively.
- ☞ Rotation is probabilistic, but heralded.

Brennen & Miyake scheme PRL 101, 010502 (2008).



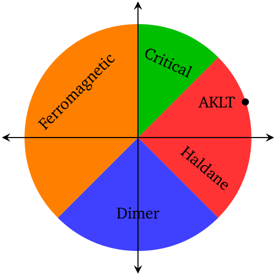
- 2D lattice of spins
- Individual chains as "quantum wires"
- Two-qubit logic (CPHASE) by dynamical coupling

Physical Implementation

Atoms or polar molecules in an optical lattice

Gap protects against thermal noise

Imperfect Hamiltonians



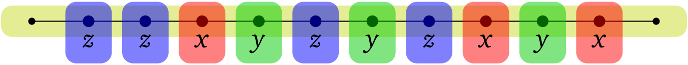
- Can only engineer H_{AKLT} to some accuracy.
- Consider the Hamiltonians

$$H(\theta) = \sum_{j=1}^{n} \cos\theta(\vec{S}_j \cdot \vec{S}_{j+1}) + \sin\theta(\vec{S}_j \cdot \vec{S}_{j+1})^2.$$

- System is gapped in the Haldane phase (Haldane's conjecture/fact)
- ➠ Computation should still be robust to noise.
- Using the same measurement scheme as for the AKLT state, how well can we perform single-qubit operations?

- Can in principle determine the output fidelity for any given mmt sequence and initial state. $\Rightarrow$ But there are an exponential number!

- Work in the gate teleportation picture (Doherty & Bartlett PRL 103, 020506 (2009).)



$$|\mathcal{G}\rangle \Rightarrow |s_1, s_2, \ldots, s_n\rangle_B \otimes \left(\sigma_{s_n}\sigma_{s_{n-1}} \cdots \sigma_{s_1}\right)_C |\Psi^-\rangle_{AC}$$

- Now only need to evaluate $\langle X_A X_C\rangle_{\vec{s}}$ and $\langle Z_A Z_C\rangle_{\vec{s}}$ for each mmt result $\vec{s}$, suitably corrected for byproducts, and average ($|\Psi^-\rangle$ gives -1 for both)

- $\langle X_A X_C\rangle' = \sum_{\vec{s}}\langle X_A X_C\rangle_{\vec{s}}$ with $\langle X_A X_C\rangle_{\vec{s}} = \langle\mathcal{G}|\, X_A\, P_{\vec{s},B}\, U^\dagger_{\vec{s},C} X_C U_{\vec{s},C}\, |\mathcal{G}\rangle$.
  $P_{\vec{s}}$ is the measurement projector, and $U_{\vec{s}}$ is the correction operator.

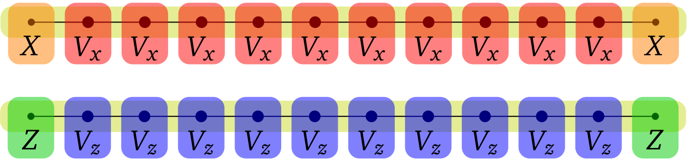
☞ Want to evaluate $\langle X_A X_C \rangle_{\vec{s}} = \langle \mathcal{G}| \, X_A \; P_{\vec{s},B} \; U^{\dagger}_{\vec{s},C} X_C U_{\vec{s},C} \; |\mathcal{G}\rangle$

- $U_{\vec{s}}$ is a Pauli operator, just $X^{\#(x\,\mathrm{or}\,y)} Z^{\#(z\,\mathrm{or}\,y)}$.
- $\left(Z^{\#(z\,\mathrm{or}\,y)} X^{\#(x\,\mathrm{or}\,y)}\right) X \left(X^{\#(x\,\mathrm{or}\,y)} Z^{\#(z\,\mathrm{or}\,y)}\right) = (-1)^{\#(z\,\mathrm{or}\,y)} X$
- $(-1)^{\#(z\,\mathrm{or}\,y)} \, |s\rangle = e^{i\pi J_x} \, |s\rangle$.
- $\langle X_A X_C \rangle_{\vec{s}} = \langle \mathcal{G}| \, X_A \, \vec{V}_{x,B} P_{\vec{s},B} X_C \, |\mathcal{G}\rangle$ for $\vec{V}_{x,B} = \left(\bigotimes_{k=1}^{n} e^{i\pi J_{x,B_k}}\right)$.

➠ Great! Now $\sum_{\vec{s}} P_{\vec{s},B} = \mathbb{1}_B$ and $\langle X_A X_C \rangle' = \langle \mathcal{G}| \, X_A \, \vec{V}_{x,B} X_C \, |\mathcal{G}\rangle$

$\langle X_A X_C \rangle' \Rightarrow$

$\langle Z_A Z_C \rangle' \Rightarrow$



- We only need to evaluate string operators $X_A \vec{V}_{x,B} X_C$ and $Z_A \vec{V}_{z,B} Z_C$.

- Similar to string operators $S_{x,j} \vec{V}_x S_{x,k}$ showing the "hidden order" of ground states in the Haldane phase
  (finite at any length, unlike usual spin correlations which decay exponentially.)

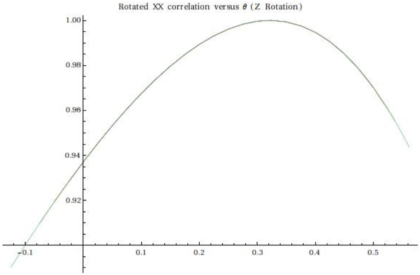
- Haldane conjecture/fact (plus fixed BCs) $\Rightarrow \langle X_A X_C \rangle' = \langle Z_A Z_C \rangle' = 1$:
  - $X_A \vec{V}_{x,B} X_C$ is a global rotation by π about $\hat{x}$;
  - H is rotationally-invariant and ground state unique.

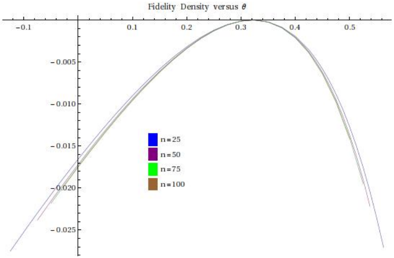
- Near-AKLT ground states also have well-defined qubit encoding.

- For single qubit rotations we need $\langle X_A R(\theta) X_C R(\theta)^\dagger \rangle'$, etc.
  Attempt rotations only a few times, average results.

Numerical results:

- ☞ Find ground states of the open/fixed chain using DMRG
  (actually use MPS form and imaginary time evolution)
- ☞ Localizeable entanglement = 1 throughout Haldane phase, as expected.
- ☞ Rotation fidelity drops off slowly with theta (no computational phase)





- Nearby Hamiltonian: $H(\theta_{\text{AKLT}} + \epsilon) \approx H_{\text{AKLT}} + \frac{5}{6}\epsilon \sum_{j=1}^{n} \text{SWAP}_{j,j+1}$
- Perhaps $|\mathcal{G}(\theta)\rangle = \alpha\,|\mathcal{G}_{\text{AKLT}}\rangle + \sum_{j=1}^{n} \beta_j \text{SWAP}_{j,j+1}\,|\mathcal{G}_{\text{AKLT}}\rangle$?
- Can we improve fidelity by using swap-invariant mmt sequences?

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- Doing nothing is already swap invariant: only # of x's, y's, z's matters.
- $R_x(\varphi)$: Action of mmt sequence $|x\rangle |z_\varphi\rangle |x\rangle \Rightarrow X(ZR_x(\varphi))X = -ZR_x(\varphi)$
  $|z_\varphi\rangle |x\rangle |x\rangle$, $|x\rangle |x\rangle |z_\varphi\rangle \Rightarrow ZR_x(\varphi))X$. Swap invariance! (up to a sign)
- For $|\mathcal{G}(\theta)\rangle$ a superposition of $|\mathcal{G}_{\text{AKLT}}\rangle$ and SWAP $|\mathcal{G}_{\text{AKLT}}\rangle$, there are different paths to same properly-rotated state, but ± 1 phase.

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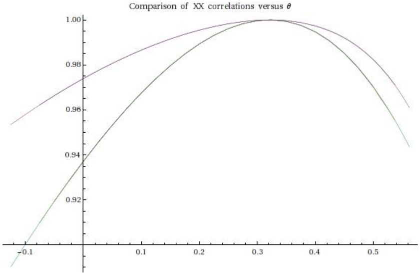
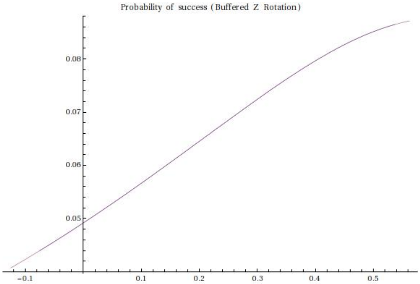
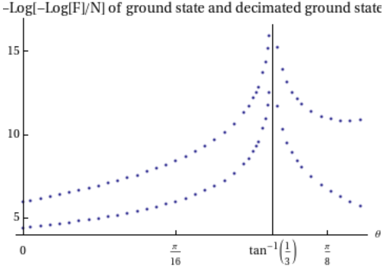
- Fidelity should improve, but overall probability of success will vary


Comparison of XX correlations versus θ
1.00
0.98
0.96
0.94
0.92
-0.1
0.1
0.2
0.3
0.4
0.5


Probability of success (Buffered Z Rotation)
0.08
0.07
0.06
0.05
-0.1
0.1
0.2
0.3
0.4
0.5

- Is there some simple transformation making the chain more AKLTlike?
  - Should have something to do with repeated mmt outcomes
  - Should respect the rotational symmetry
- Project neighbors onto spin-zero — $|\mathcal{G}_{\mathrm{AKLT}}\rangle$ invariant!
  - $|S=0\rangle = \frac{1}{\sqrt{3}}\left(|xx\rangle - |yy\rangle + |zz\rangle\right)$
  - $|\mathcal{G}\rangle = \sum_{\{s_k\}} |s_1, s_2, \ldots, s_n\rangle_B \otimes (\sigma_{s_n}\sigma_{s_{n-1}}\cdots\sigma_{s_1})_C |\Psi^-\rangle_{AC}$



- Project neighbors onto spin 0 or 1
- Project n-n-neighbors onto spin-zero (gate buffering)
- Probably many more
- Dimer state also a fixed-point.

Summary and Outlook

- ☞ Can efficiently characterize computational ability of spin-1 chains.
- ☞ Near-AKLT ground states make good quantum wires.
- ☞ Buffered measurements improve performance.
- ☞ Simple projection operations make AKLTlike states even more so.

- ☞ Repeated gates: Markovian noise?
- ☞ Two-qubit gates (CPHASE)? Thermal noise?
- ☞ Analytic results (perturbative or variational using swap states)?
- ☞ How does buffering relate to renormalization?

Thanks for your attention!