Part I

Chapter 2

2 *Show that the two definitions of continuity are equivalent:*

- *Topological definition: f is continuous if for any open $U \subset Y$, $f^{-1}(U) \subset X$ is also open*
- *$\epsilon - \delta$ definition: f is continuous when for all x and $\epsilon > 0$, there exists a $\delta > 0$ such that $||x - x'|| < \delta$ implies $||f(x) - f(x')|| < \epsilon$.*

$\Rightarrow$ Pick $f(x) \in Y$ and $\epsilon > 0$. Set $U = \{y : ||y - f(x)|| < \epsilon\}$, which is an open set . Thus $f^{-1}(U)$ is also an open set. This means that for all $x \in f^{-1}(U)$ all points sufficiently close to x are also in $f^{-1}(U)$ or, put differently, for all x there exists $\delta > 0$ such that $N_\delta(x) = \{x' : ||x - x'|| < \delta\} \subset f^{-1}(U)$. Thus, $f(x') \in U$ for all $x' \in N_\delta(x)$, which means there exists a $\delta > 0$ such that $||x - x'|| < \delta$ implies $||f(x) - f(x')|| < \epsilon$

$\Leftarrow$ Pick any open U. Now we must show that $f^{-1}(U)$ is open. For any $f(x) \in U$ there exists $\delta > 0$ such that $||x - x'|| < \delta$ means $||f(x) - f(x')|| < \epsilon$, so we can pick ϵ small enough to ensure that $\{f(x') : ||f(x) - f(x')|| < \epsilon\} \subset U$. Hence, for any point $x \in f^{-1}(U)$, points sufficiently close are also in $f^{-1}(U)$: $f^{-1}(U)$ is open.

3 *Show that $S^n \subset \mathbb{R}^{n+1}$ with its induced topology is a manifold.*

We need to check that the transition functions are smooth. This can be ensured by mapping each of the subsets of the sphere back into $\mathbb{R}^{n+1}$ and applying the transition function there (it's the identity function there).

4: *Show that if M is a manifold and U is an open subset of M, then U with its induced topology is a manifold.*

The only thing one really needs to check is the smoothness of the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$. But this property is directly inherited from M, since subsets of the form $U \cap S$ where S is open in M are themselves open subsets of M.

Chapter 3

7: *Show that $v + w$ and $gw \in \mathrm{Vect}(M)$.*

$$[v + w](f + g) = v(f + g) + w(f + g) = v(f) + w(f) + v(g) + w(g) = [v + w](f) + [v + w](g)$$
$$[v + w](\alpha f) = v(\alpha f) + w(\alpha f) = \alpha(v(f) + w(f)) = \alpha[v + w](f)$$
$$[v + w](fg) = v(fg) + w(fg) = v(f)g + fv(g) + w(f)g + fw(g) = [v + w](f)g + f[v + w](g).$$

$$[gw](f + h) = g\,w(f + h) = g(w(f) + w(h)) = [gw](f) + [gw](h).$$
$$[gw](\alpha f) = g\,w(\alpha f) = \alpha g\,w(f) = \alpha[gw](f).$$
$$[gw](fh) = g\,w(fh) = g(w(f)h + fw(h)) = g\,w(f)h + fg\,w(h) = [gw](f)\,h + f\,[gw](h).$$

8: *Show that the following rules hold for all $v, w \in \mathrm{Vect}(M)$ and $f, g \in C^\infty(M)$: 1. $f(v+w) = fv + fw$, 2. $(f+g)v = fv + gv$, 3. $(fg)v = f(gv)$, 4. $1v = v$, where 1 denotes the constant function equal to 1 on all of M. This makes $\mathrm{Vect}(M)$ a* module *over $C^\infty(M)$.*

$f(v+w)[g] = fv(g) + fw(g)$, so the first one holds. The remainder work exactly the same way: apply both sides of the equation to a test function and show that they are equal.

9: *Show that if $v^\mu \partial_\mu = 0$ we must have $v^\mu = 0$ for all μ.*

Choose as test functions the coordinate functions x^μ so that $\partial_\mu x^\mu = 1$ for all μ, meaning $v = 0$.

10: *Let $v, w \in \mathrm{Vect}(M)$. Show that $v = w$ iff $v_p = w_p$ for all $p \in M$.*

If $v = w$ then $v_p = w_p$ since $v_p(f) = v(f)[p] = w(f)[p] = w_p$ for all f. If $v_p = w_p$, then $v(f)[p] = v_p(f) = w_p(f) = w(f)[p]$ for all $p \in M$ and $f \in C^\infty(M)$, so $v = w$.

11: *Show that T_pM is a vector space over the real numbers.*

* Need to show linearity, closedness, and existence of a zero.

12: *Check that $\gamma'(t) \in T_{\gamma(t)}M$.*

$\gamma'(t)[f+g] = \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)) + g(\gamma(t))) = \gamma'(t)[f] + \gamma'(t)[g]$. Similarly we get $\gamma'(t)[\alpha f] = \alpha\gamma'(t)[f]$. Finally, $\gamma'(t)[fg] = \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t))g(\gamma(t))) = \gamma'(t)[f]g + f\gamma'(t)[g]$.

13: *Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $\phi(t) = e^t$. Let x be the usual coordinate function on $\mathbb{R}$. Show that $\phi^* x = e^x$.*

The coordinate of the transformed point $\phi(p)$ is clearly $e^{x(p)}$, so $\phi^* x$ should be e^x. More formally, $\phi^* x(p) = x(\phi(p)) = x(e^p) = e^{x(p)}$ so indeed $\phi^* x = e^x$.

14: *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation counterclockwise by an angle θ. Let x, y be the usual coordinate functions on $\mathbb{R}^2$. Show that $\phi^* x = x\cos\theta - y\sin\theta$ and $\phi^* y = x\sin\theta + y\cos\theta$.*

Again $\phi^* x(p) = x(\phi(p))$, the x coordinate of the rotated point. If p is the point (s,t), then $\phi(p) = (s', t') = (s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)$. Thus $\phi^* x(p) = s\cos\theta - t\sin\theta = x(p)\cos\theta - y(p)\sin\theta$ and similarly for $\phi^* y$. Since the point p is arbitrary, the desired result follows.

15: *$\phi : M \to N$ is smooth if $f \in C^\infty(N)$ implies $\phi^* f \in C^\infty(M)$. Show that this definition is consistent with smooth functions $f : M \to \mathbb{R}$ and smooth curves $\gamma : \mathbb{R} \to M$.*

A function $f : N \to \mathbb{R}$ is smooth if $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for all α. Pulling the function back by using ϕ makes a new smooth function $f' : M \to \mathbb{R}$ where $f' = f \circ \phi$ since its smoothness is determined by that of $(f \circ \phi) \circ (\varphi_\alpha \circ \phi)^{-1} = f \circ \varphi_\alpha^{-1}$ which is assumed to be smooth. Any other coordinate system for M would also produce a smooth result, since the different systems are related by smooth coordinate transformations.

16: *Prove that $(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t))$*

$\phi_*(\gamma'(t))[f] = \gamma'(t)(\phi^* f) = \gamma'(t)(f \circ \phi) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi)(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\phi(\gamma(t))) = \frac{\mathrm{d}}{\mathrm{d}t} f((\phi \circ \gamma)(t)) = (\phi \circ \gamma)'(t)[f]$

17: Show that the pushforward operation $\phi_* : T_pM \to T_{\phi(p)}N$ is linear.

$\phi_*(v+w)[f] = (v+w)(\phi^* f) = v(\phi^* f_+ w(\phi^* f) = \phi_* v[f] + \phi_* w[f]$ and similarly for scalars α.

18: *Show that if $\phi : M \to N$ we can push forward a vector field v on M to obtain a vector field $\phi_* v$ on N satisfying $(\phi_* v)_q = \phi_*(v_p)$ whenever $\phi(p) = q$.*

Since the vector fields can be specified pointwise, it suffices to consider a general point p. Now, $(\phi_* v)_q[f] = \phi_* v(f)[q] = v(f \circ \phi)[p] = v_p(\phi^* f) = \phi_*(v_p)[f]$.

19: *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by an angle θ. Let ∂_x, ∂_y be the coordinate vector fields on $\mathbb{R}^2$. Show that at any point of $\mathbb{R}^2$: $\phi_* \partial_x = (\cos\theta)\partial_x + (\sin\theta)\partial_y$ and $\phi_* \partial_y = -(\sin\theta)\partial_x + (\cos\theta)\partial_y$.*

Intuitively, this is just a rotation of the vectors, since they are tangent vectors to the coordinate paths which are rotated by ϕ. To make this more precise, consider $(\phi_* \partial_x) f = \partial_x(\phi^* f) = \partial_x(f \circ \phi)$. Note that ∂_x really means the derivative of the function with respect to whatever quantity is in the first input; choosing the point (x, y) as the input makes this automatic. Then $\partial_x(f \circ \phi)(x, y) = \partial_u f \cdot \partial_x u + \partial_v f \cdot \partial_x v$, where $(u, v) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = \phi(x, y)$. We then obtain $(\phi_* \partial_x) f = (\cos\theta)\partial_u f + (\sin\theta)\partial_v f$. Since u is the first input to f, ∂_u is the same as ∂_x in the rotated space, and similarly for v and y. This gives us $\phi_* \partial_x = (\cos\theta)\partial_x + (\sin\theta)\partial_y$; the same argument works for $\phi_* \partial_y$.

20: *Let v be the vector field $x^2\partial_x + y\partial_y$ on $\mathbb{R}^2$. Calculate the integral curves $\gamma(t)$ and see which ones are defined for all t.*

We start by making up a representation. $\gamma'(t)[f] = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \partial_\mu f \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}$, so $\gamma'(t) = \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t}\partial_\mu$. If $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = \dot{x}(t)\partial_x + \dot{y}(t)\partial_y$. Then $\dot{x} = x^2$ and $\dot{y} = y$. The solution is $\gamma(t) = (x(0)/(1 - x(0)t), y(0)e^t)$, which is finite for finite t provided $1 - x(0)t \neq 0$. Thus, only the curves with $x(0) = 0$ are defined for all t.

21: *Show that ϕ_0 is the identity map id:$X \to X$, and that for all $s, t \in \mathbb{R}$ we have $\phi_t \circ \phi_s = \phi_{t+s}$.*

The point p is translated to the point $\gamma(t)$ along the integral curve γ of v with p as the initial point. Since $\gamma(0) = p$, $\phi_0(p) = p$. Moreover, as the integral curve is the solution to a first-order differential equation, the solutions are unique and the curves don't intersect. If we move apply $\phi_t \circ \phi_s$ to p we are first moving the point $p = \gamma(0)$ to $\gamma(s)$ and then using this point as the new intial value for the integral curve so that we can move it to $\widetilde{\gamma}(t)$. But the curves are unique and any point on the curve defines a boundary condition which together with the differential equation, yields the curve. Thus, γ and $\widetilde{\gamma}$ are the same curve with the same parameterization (except for the offset).

22: *Consider the normalized vector fields in the r and θ direction on the plane in polar coordinates (not defined at the origin):*

$$v = \frac{x\partial_x + y\partial_y}{\sqrt{x^2 + y^2}}, \qquad w = \frac{x\partial_y - y\partial_x}{\sqrt{x^2 + y^2}}$$

Calculate $[v, w]$.

Let's put this into component notation first. Define $\bar{r} = 1/\sqrt{x^2+y^2}$, $x_0 = x$, and $x_1 = y$. Then $v = \bar{r}x^k\partial_k$ while $w = \bar{r}\epsilon_i^j x^i\partial_j$, for ϵ_i^j the antisymmetric symbol/tensor. Before getting started, note that $\partial_j\bar{r} = -x_j\bar{r}^3$ (watch out for the lower index!). Now

$$\begin{aligned}
[v,w] &= vw - wv = \bar{r}x^k\partial_k(\bar{r}\epsilon_i^j x^i\partial_j) - \bar{r}\epsilon_i^j x^i\partial_j(\bar{r}x^k\partial_k) \\
&= \bar{r}^2\epsilon_i^j x^k\left(\delta_k^i\partial_j + x^i\partial_k\partial_j - \bar{r}^2 x_k x^i\partial_j\right) - \bar{r}^2\epsilon_i^j x^i\left(\partial_j + x^k\partial_j\partial_k - \bar{r}^2 x_j x^k\partial_k\right) \\
&= -\bar{r}^4 x^k x_k\epsilon_i^j x^i\partial_j = -\bar{r}^2\epsilon_i^j x^i\partial_j = -\frac{x\partial_y - y\partial_x}{x^2+y^2} = -\frac{w}{r},
\end{aligned}$$

since $[\partial_i, \partial_j] = 0$ and $\epsilon_i^j x^i x_j = 0$.

23 *Check that* $[v,w](f)(p) = \frac{\partial^2}{\partial t\partial s}\left[f(\psi_s(\phi_t(p))) - f(\phi_t(\psi_s(p)))\right]_{s=t=0}$

Start with $(vf)(p) = \frac{d}{dt}f(\phi_t(p))|_{t=0}$ and $(wf)(p) = \frac{d}{ds}f(\psi_s(p))|_{s=0}$. Then $(vwf)(p) = v(wf)(p) = \frac{d}{dt}wf(\phi_t(p))|_{t=0} = \frac{\partial^2}{\partial t\partial s}f(\psi_s(\phi_t(p)))|_{s=t=0}$. Note that wf means take f and displace its input a little bit, and evaluate the derivative at zero displacement. In the compound expression, the input to f is $\phi_t(p)$, so ψ_s ends up first inside f in the final expression. Alternatively, we could work from inside-out: $v(wf)(p) = v\left(\frac{d}{dt}f(\psi_s(\cdot))|_{s=0}\right)(p) = \frac{\partial^2}{\partial t\partial s}f(\psi_s(\phi_t(p)))|_{s=t=0}$. Similarly, $(wvf)(p) = \frac{\partial^2}{\partial t\partial s}f(\phi_t(\psi_s(p)))|_{s=t=0}$, so the expression for the commutator holds.

24 *Show that for all vector fields u, v, w on a manifold, and all real numbers α and β, we have:*

1. $[v,w] = -[w,v]$
2. $[u, \alpha v + \beta w] = \alpha[u,v] + \beta[u,w]$
3. *The* **Jacobi identity***:* $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$

1. $[u,v] = uv - vu = -(vu - uv) = -[v,u]$
2. $[u,\alpha v + \beta w] = u(\alpha v + \beta w) - (\alpha v + \beta w)u = \alpha[u,v] + \beta[u,w]$
3. $[u,[v,w]] = u(vw-wv)-(vw-wv)u$, so $[u,[v,w]]+[v,[w,u]]+[w,[u,v]] = u(vw-wv) - (vw-wv)u + v(wu-uw) - (wu-uw)v + w(uv-vu) - (uv-vu)w = 0$

Chapter 4

25 *Show that $\omega + \mu$ and $f\omega$ are really 1-forms, i.e. show linearity over $C^\infty(M)$.*

That ω and μ are 1-forms means they are (separately) linear over $C^\infty(M)$. So $[\omega+\mu](v+w) = \omega(v+w) + \mu(v+w) = \omega(v) + \omega(w) + \mu(v) + \mu(w) = [\omega+\mu](v) + [\omega+\mu](w)$. $f\omega$ is just multiplication by f of the result of applying ω, so $[f\omega](v+w) = f\omega(v+w) = f\omega(v) + f\omega(w)$.

26 *Show that $\Omega^1(M)$ is a module over $C^\infty(M)$.*

There are four conditions to check. Let $f, g \in C^\infty(M)$ and $\omega, \mu \in C^\infty(M)$. The first condition is $f(\omega+\mu) = f\omega + f\mu$, which is true by applying each side of the equation to a test vector v and using the previous exercise to show the two resulting expressions are equal. Same for the conditions $(f+g)\omega = f\omega + g\omega$, $(fg)\omega = f(g\omega)$, and $1\omega = omega$. Basically the point is that by applying the 1-forms to test vectors, everything turns into functions and the equality of the different expressions follows immediately.

27 *Show that* $d(f+g) = df + dg$, $d(\alpha f) = \alpha\, df$, $(f+g)dh = f\, dh + g\, dh$, *and* $d(fg) = f(dg) + g(df)$ *for any* $f, g, h \in C^\infty(M)$ *and any* $\alpha \in \mathbb{R}$.

The differential map d is defined by $df(v) = vf$. Applying this definition to the first equation, we obtain $d(f+g)v = v(f+g) = vf + vg = df(v) + dg(v) = [df + dg](v)$ as intended. Similarly, $d(\alpha f)(v) = v(\alpha f) = \alpha v f = \alpha df(v)$ and $(f+g)dh(v) = (f+g)vh = f\, vh + g\, vh = [f\, dh + g\, dh](v)$. For the last condition, compute $d(fg)(v) = v(fg)$. By the definition of v as a vector field, $v(fg) = gv(f) + fv(g)$, so we obtain $d(fg)(v) = [f(dg) + g(df)](v)$.

28 *Suppose* $f(x^1, \ldots, x^n)$ *is a function on* $\mathbb{R}^n$. *Show that* $df = \partial_\mu f\, dx^\mu$.

Consider an arbitrary vector field $v = v^\nu \partial_\nu$. Then $df(v) = vf = v^\nu \partial_\nu f$. On the other hand, $(\partial_\mu f\, dx^\mu)v = \partial_\mu f\, v x^\mu = \partial_\mu f\, v^\nu \partial_\nu x^\mu = \partial_\mu f\, v^\nu \delta^\mu_\nu = v^\nu \partial_\nu f$. So $df = \partial_\mu f\, dx^\mu$ as intended.

29 *Show that the 1-forms* $\{dx^\mu\}$ *are linearly independent.*

Suppose $\omega = \omega_\mu\, dx^\mu = 0$. Then $\omega(\partial_\nu) = \omega_\mu\, dx^\mu(\partial_\nu) = \omega_\mu \partial_\nu x^\mu = \omega_\nu = 0$.

30 *Show that* $\omega(v)(p)$ *really depends only on* v_p, *not on the values of* v *at other points. Also, show that a 1-form is determined by its values at points. In other words, if* ω, ν *are two 1-forms on* M *with* $\omega_p = \nu_p$ *for every point* $p \in M$, *then* $\omega = \nu$.

Suppose w is a vector field for which $w_p = 0$ but generally $w_q \neq 0$. By linearity, $\omega(v + w) = \omega(v) + \omega(w)$. Specializing to the point p, we have $\omega(v + w)(p) = \omega(v)(p) + \omega(w)(p) = \omega_p(v_p) + \omega_p(w_p)$. But since $w_p = 0$, the latter term is zero, and $\omega(v + w)(p) = \omega(v)(p)$. This means we can change the field v at points distinct from p and still obtain the same action by *omega* at the desired point. Note that this works due to the linearity of the map ω.

Since we can "decompose" the field v into tangent vectors v_p at each point $p \in M$, (excercise 10), we have $\omega_p(v_p) = \nu_p(v_p) = \omega(v)(p) = \nu(v)(p)$. This has to hold for any v and all p, so $\omega = \nu$.

31 *Show that the dual of the identity map on a vector space* V *is the identity map on* V^*. *Suppose that we have linear maps* $f : V \to W$ *and* $g : W \to X$. *Show that* $(gf)^* = f^* g^*$.

The dual of a function f is defined by $(f^*\omega)(v) = \omega(f(v))$. So if f is the identity, then $(f^*\omega)(v) = \omega(v)$, meaning that f^* must be the identity as well. Now, $[(gf)^*\omega](v) = \omega(g(f(v))) = [g^*\omega](f(v)) = [f^* g^* \omega](v)$, so $(gf)^* = f^* g^*$.

32 *Show that the pullback defined by the equation* $(\phi^*\omega)_p = \phi^*(\omega_q)$, *where* $\phi(p) = q$, *really exists and is unique.*

Start with $(\phi^*\omega)_p(v_p) = \phi^*\omega(v)(p) = \omega(\phi_* v)(\phi(p)) = \omega(\phi_* v)(q)$. The other side of the equation gives $\phi^*(\omega_q)v_q = \omega_q(\phi_* v_q)$, the same. Uniqueness follows from linearity: if there were two possible outputs, we could consider their difference. By linearity this would be the result of applying the map to the difference of the fixed input, which is zero. But the output of zero is zero, so the two possible outputs are equal.

33 *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be given by* $\phi(t) = \sin t$. *Let* dx *be the usual 1-form on* $\mathbb{R}$. *Show that*

$\phi^* dx = \cos t\, dt$.

$\phi^*(df) = d(\phi^* f)$ since the exterior derivative is natural. Thus $\phi^*(df) = d(\sin t) = \cos t\, dt$.

34 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation counterclockwise by the angle θ. Let dx, dy be the usual basis of 1-forms on $\mathbb{R}^2$. Show that $\phi^* dx = \cos\theta\, dx - \sin\theta\, dy$ and $\phi^* dy = \sin\theta\, dx + \cos\theta\, dy$.*

$\phi^* dx = d(\phi^* x) = d(x\cos\theta - y\sin\theta) = \cos\theta\, dx - \sin\theta\, dy$. Similarly for $\phi^* dy$.

35 *Show that the coordinate 1-forms dx^μ really are the differentials of the local coordinates x^μ on U.*

Technically, x^μ is the local coordinate on $\mathbb{R}^n$ which picks out the μth component. When speaking of x^μ on U, we really mean $\phi^* x^\mu$, where ϕ is the map from U to $\mathbb{R}^n$. Thus dx^μ is really $d(\phi^* x^\mu) = \phi^* dx^\mu$, so the sloppiness of the notation doesn't injure the relationship between functions and 1-forms.

36 *Show that $dx'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$. Show that for any 1-form ω on $\mathbb{R}^n$, writing $\omega = \omega_\mu dx^\mu = \omega'_\nu dx'^\nu$, the components ω'_ν are related to the components ω_μ by $\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu$.*

For the first equation, apply both sides to the partial ∂_α: $dx'^\nu(\partial_\alpha) = \partial_\alpha(x'^\nu)$ and $\frac{\partial x'^\nu}{\partial x^\mu} dx^\mu(\partial_\alpha) = \frac{\partial x'^\nu}{\partial x^\mu}\partial_\alpha(x^\mu) = \frac{\partial x'^\nu}{\partial x^\mu}\delta^\mu_\alpha = \frac{\partial x'^\nu}{\partial x^\alpha} = \partial_\alpha(x'^\nu)$. For the second, use the same trick. Apply ω to $\frac{\partial}{\partial x'^\alpha}$. The left expression gives $\omega_\mu dx^\mu(\frac{\partial}{\partial x'^\alpha}) = \omega_\mu \frac{\partial x^\mu}{\partial x'^\alpha}$, while the right expression gives ω'_α.

37 *Show that $\phi^*(dx'^\nu) = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$.*

$\phi^*(dx'^\nu)\partial_\mu = d(\phi^* x'^\nu)\partial_\mu = \partial_\mu(\phi^* x'^\nu) = \frac{\partial x'^\nu}{\partial x^\mu}$.

38 *Let $e_\mu = T^\nu_\mu \partial_\nu$, where ∂_ν are the coordinate vector fields associated to local coordinates on an open set U and T^ν_μ are functions on U. Show that the vector fields e_μ are a basis of vector fields on U iff for each $p \in U$ the matrix $T^\nu_\mu(p)$ is invertible.*

- $\Rightarrow$ The matrix $T^\nu_\mu(p)$ is invertible, so $\det T \neq 0$, meaning the e_u are linearly independent at p and thus form a basis at p. Since this works for all p, the vector fields e_μ are a basis.
- $\Leftarrow$ Try to expand an arbitrary ω in terms of the e_μ: $\omega = \omega^\mu e_\mu$. We can also express ω in terms of the ∂_ν as $\omega = \omega'^\nu \partial_\nu$, with $\omega'^\nu = T^\nu_\mu \omega^\mu$. If T were not invertible, we would not be able to construct the ω^μ from the ω'^ν (which are the ones guaranteed to exist), so e_μ wouldn't be a basis. Since it is a basis, T must be invertible.

39 *Use the previous exercise to show that the dual basis exists and is unique*

Let f^μ be the putative dual basis to the e_ν. Then $f^\mu(e_\nu) = \delta^\mu_\nu = T^\lambda_\nu f^\mu \partial_\lambda$. Now suppose $f^\mu = S^\mu_\lambda dx^\lambda$. Inserting into the previous equation we obtain $T^\lambda_\nu S^\mu_\lambda = \delta^\mu_\nu$, so S is the inverse of T. Since T is invertible, the f^μ are well-defined and unique.

40 *Let e_μ be a basis of vector fields on U and let f^μ be the dual basis of 1-forms. Let $e'_\mu = T^\nu_\mu e_\nu$ be another basis of vector fields, and let f'^μ be the corresponding dual basis of 1-forms. Show that $f'^\mu = (T^{-1})^\mu_\nu f^\nu$. Show that if $v = v^\mu e_\mu = v'^\mu e'_\mu$, then $v'^\mu = (T^{-1})^\mu_\nu v^\nu$ and that if $\omega = \omega_\mu f^\mu = \omega'_\mu f'^\mu$ then $\omega'_\mu = T^\nu_\mu \omega_\nu$.*

This is where the "historical" definitions of co- and contra-variant come from, specifying whether something transforms like the components of a vector or like the basis vectors themselves. (Vectors, being susceptible to pushforward, are covariant. 1-forms are contravariant, but the components of a vector are also contravariant, while the components of 1-forms are covariant.) For f'^{μ}, apply it to e'_ν: $f'^{\mu}(e'_\nu) = \delta^\mu_\nu$. Using the definition of e'_ν, $\delta^\mu_\nu = T^\lambda_\nu f'^\mu(e_\lambda)$. This works if $f'^\mu = (T^{-1})^\mu_\nu f^\nu$. All this can be done more easily in matrix notation. For the next case, let $\mathbf{v}$ be the ordered set of v^μ and likewise $\mathbf{e}$ the set of e_μ. Then $v = \mathbf{v} \cdot \mathbf{e}$. Meanwhile, $\mathbf{e}' = T\mathbf{e}$, so clearly we're going to need $\mathbf{v}' = (T^{-1})^T\mathbf{v}$ so that $\mathbf{v}' \cdot \mathbf{e}' = \mathbf{v}T^{-1}T\mathbf{e} = v$. Thus, we predict that $v'^\mu = (T^{-1})^\mu_\nu v^\nu$, using the inverse and transposed T as given in the definition of e'_μ. Similarly $\omega = \omega \cdot \mathbf{f}$, so $\omega'_\mu = T^\nu_\mu \omega_\nu$.

41 *Show that* $u \wedge v \wedge w = \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} dx \wedge dy \wedge dz$. *Compare this to* $\vec{u} \cdot (\vec{v} \times \vec{w})$.

Algebra.

42 *Show that if* a, b, c, d *are four 1-forms in a 3-dimensional space, then* $a \wedge b \wedge c \wedge d = 0$.

Since there are only three linearly independent 1-forms in this space, the 4-fold wedge product will always contain two of the basis elements twice, which by antisymmetry, means the entire expression must be zero.

43 *Describe* $\bigwedge V$ *if* V *is 1-,2-,3-, or 4-dimensional.*

1-dimensional: scalars and vectors. 2-dimensional: scalars, vectors, and areas. 3-dimensional: scalars, vectors, areas, and volumes. 4-dimensional: all the previous, plus 4-volumes.

44 *Let* V *be an* n*-dimensional vector space. Show that* $\bigwedge^p V$ *is empty for* $p > n$ *and that for* $0 \leq p \leq n$ *the dimension of* $\bigwedge^p V$ *is* $n!/p!(n-p)!$.

As in exercise 42, elements of $\bigwedge^p V$ are wedge products of p different terms, but since there are only n independent possibilities, $\bigwedge^p V$ is empty for $p > n$. Wedging p different terms for $p \leq n$ can be done in $\binom{n}{p}$ ways since by antisymmetry the order of the terms can only affect the sign of the result.

45 *Show that* $\bigwedge V = \bigoplus_p \bigwedge^p V$.

Since we've defined $\bigwedge V$ formally, as the linear space of wedge products of elements of V, we then immediately have that the different sectors corresponding to different number of wedgings p are disjoint. Thus we can treat these completely independently from one another and form n-tuples containing entries from each sector. This will be an element of $\bigwedge V$ since adding elements of distinct grade is also purely formal and can always be resolved into the various graded components. Thus, $\bigwedge V = \bigoplus_p \bigwedge^p V$.

46 *Given a vector space* V*, show that* $\bigwedge V$ *is a graded commutative or supercommutative algebra; that is, if* $\omega \in \bigwedge^p V$ *and* $\mu \in \bigwedge^q V$*, then* $\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega$.

This is essentially just a counting argument. ω is the wedge of p things, μ q. So to invert the order of wedging them together means transporting p elements of V past q elements of V, each time picking up a minus one.

47 *Show that differential forms are contravariant. That is, show that if $\phi : M \to N$ is a map from the manifold M to the manifold N, there is a unique pullback map $\phi^* : \Omega(N) \to \Omega(M)$ agreeing with the usual pullback on 0-forms (functions) and 1-forms, satisfying $\phi^*(\alpha\omega) = \alpha\phi^*\omega$, $\phi^*(\omega + \mu) = \phi^*\omega + \phi^*\mu$ and $\phi^*(\omega \wedge \mu) = \phi^*\omega \wedge \phi^*\mu$, for all $\omega, \mu \in \Omega(N)$ and $\alpha \in \mathbb{R}$.*

I take the three conditions to almost be the definition of the map. For a general ω, resolve it into a linear combination of graded components using an arbitrary basis of 1-forms to generate the p-forms. The by condition two the pullback applies to each term separately. By condition one the coefficients of each term don't interfere with the pullback. Finally, by condition three we apply the pullback to each of the 1-forms making up the basis p-forms, so the map is well-defined. Its unique because the pullback applied to 1-forms and functions is unique.

48 *Compare the transformation properties of 1-forms and 2-forms on $\mathbb{R}^3$ under parity. That is, let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the map $P(x, y, z) = (-x, -y, -z)$, known as the 'parity transformation'. Note that P maps right-handed bases to left-handed bases and vice versa. Compute $\phi^*\omega$ when ω is the 1-form $\omega_\mu dx^\mu$ and when it is the 2-form $\frac{1}{2}\omega_{\mu,\nu} dx^\mu \wedge dx^\nu$.*

If prime variables refer to the output space, for instance $x' = -x$, the essential point is that $\phi^*(dx'^\mu) = d(\phi^* x'^\mu) = d(-x^\mu) = -dx^\mu$. Thus $\phi^*\omega^{(1)} = -\omega$ and $\phi^*\omega^{(2)} = \omega^{(2)}$.

49 *Show that on $\mathbb{R}^n$ the exterior derivative of any 1-form is given by $d(\omega_\mu dx^\mu) = \partial_\nu\omega_\mu dx^\nu \wedge dx^\mu$.*

$d(\omega_\mu dx^\mu) = d\omega_\mu \wedge dx^\mu = \partial_\nu\omega_\mu dx^\nu \wedge dx^\mu$.

Chapter 5

50 *Show that any 2-form F on $\mathbb{R} \times S$ can be uniquely expressed as $B + E \wedge dt$ in such a way that for any local coordinates x^i on S we have $E = E_i dx^i$ and $B = \frac{1}{2}B_{ij} dx^i \wedge dx^j$.*

For some patch $U \subset S$, we can use the local coordinates x^i along with $t \in \mathbb{R}$ to define an element of $\mathbb{R}^{n+1}$. Any 2-form F can be written as $F = F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ where the summation runs over $0 \dots n$ and $x^0 = t$. We can also express this as $F = F_{0i} dt \wedge dx^i + F_{i0} dx^i \wedge dt + F_{ij} dx^i \wedge dx^j$. But then by antisymmetry of the wedge product, we might as well have $F_{i0} = -F_{0i}$. Then defining $E_i = 2F_{0i}$ we have $F = E_i dx^i \wedge dt + F_{ij} dx^i \wedge dx^j$. With $B_{ij} = 2F_{ij}$ we obtain $F = E \wedge dt + B$. This decomposition is unique because if it were true using different functions E'_i and B'_{ij}, then their difference would be zero.

51 *Show that for any form ω on $\mathbb{R} \times S$ there is a unique way to write $d\omega = dt \wedge \partial_t\omega + d_S\omega$ such that for any local coordinates x^i on S, writing $t = x^0$, we have $d_S\omega = \partial_i\omega_I dx^i \wedge dx^I$ and $dt \wedge \partial_t\omega = \partial_0\omega_I dx^0 \wedge dx^I$.*

First write $\omega = \omega_I dx^I$ for I a multi-index of length p (i.e. a p-tuple whose entries range over $0 \dots n$, where n is the dimension of S). Then $d_S\omega = \partial_i\omega_I dx^i \wedge dx^I$ and $\partial_t\omega = \partial_0\omega_I$. Again, uniqueness follows from linearity: if there were two possibilities, their difference would be zero by the linearity of d, implying their equality.

52 *Use the nondegeneracy of the metric to show that the map from V to V^* given by $v \mapsto g(v, \cdot)$ is an isomorphism, that is, 1-to-1 and onto.*

Nondegeneracy of the metric means that if $g(v,w) = 0$ for all w, then $v = 0$. Now, for the map to be an isomorphism, we have to check the two conditions. First 1-to-1: different inputs result in different outputs. So we check if the difference of the outputs is zero: $g(v,\cdot) - g(w,\cdot) = g(v-w,\cdot) \neq 0$ unless $v = w$. Next is the onto condition: is every element of V^* the image of some $v \in V$? Yes, consider $\omega \in V^*$ and consider the action on a basis: $\omega_\mu = \omega(e_\mu)$. Setting $\omega_\mu = g(v, e_\mu)$, we have to solve for v. Expanding $v = v^\mu e_\mu$ we obtain $\omega_\mu = v^\nu g(e_\nu, e_\mu) = v^\nu g_{\nu,\mu}$. But the nondegeneracy of g implies that $g_{\nu,\mu}$ is invertible. Suppose we have $g(v,w) = g_{\mu,\nu}v^\mu w^\nu = 0$ for all w. Then $g_{\mu,\nu}v^\mu = 0$ for all ν. But this only occurs when $v^\mu = 0$ for all μ, meaning that $g_{\mu,\nu}$ never annihilates an input vector; it has full rank. Thus, it's invertible. So we can solve the equation for v.

There's a more direct proof for the vector spaces we are considering here: by exercise 28, we know the dual space has the same dimension as the original space. Thus, nothing in the dual can be outside the image of the original space, since if were, it would consititute a new basis element.

53 *Let $v = v^\mu e_\mu$ be a vector field on a chart. Show that the corresponding 1-form $g(v,\cdot)$ is equal to $v_\nu f^\nu$, where f^ν is the dual basis of 1-forms and $v_\nu = g_{\mu,\nu}v^\mu$.*

We can follow the lines of the previous exercise to derive the form, or simply show that the equation is correct. Choosing an arbitrary vector field w, $g(v,w) = g_{\mu,\nu}v^\mu w^\nu$ which is equal to $v_\nu f^\nu(w) = v_\nu w^\nu$.

54 *Let $\omega = \omega_\mu f^\mu$ be a 1-form on a chart. Show that the corresponding vector field is equal to $\omega^\nu e_\nu$ where $\omega^\nu = g^{\mu,\nu}\omega_\mu$.*

For arbitrary v, $\omega(v) = \omega_\mu f^\mu(v) = \omega_\mu v^\mu$. But there should be a w such that $g(w,v) = \omega_\mu v^\mu$. Using the series expression for v and w, we obtain $\omega_\nu v^\nu = g_{\mu,\nu}w^\mu v^\nu$. This should hold for arbitrary v, so $\omega_\nu = g_{\mu,\nu}w^\mu$, or $w^\mu = g^{\mu,\nu}\omega_\nu$.

55 *Let η be the Minkowski metric on $\mathbb{R}^4$. Show that its components in the standard basis are $\eta_{\mu,\nu} = \delta_{\mu,\nu}(1-2\delta_{\mu,0})$.*

The Minkowski metric is defined for two vectors v and w as $\eta(v,w) = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3$. The form of $\eta_{\mu,\nu}$ follows immediately.

56 *Show that $g^\mu_\nu = \delta^\mu_\nu$.*

If we start with $g_{\lambda,\nu}$ we can raise the first index with the metric, obtaining $g^\mu_\nu = g^{\mu,\lambda}g_{\lambda,\nu}$. But $g^{\lambda,\nu}$ is the matrix inverse of $g_{\lambda,\nu}$, so the result of the product is the Kronecker delta.

57 *Show that the inner product of p-forms is nondegenerate by supposing that $(e^1,\ldots,e^n)$ is any orthonormal basis of 1-forms in some chart, with $g(e^i,e^i) = \epsilon(i)$ for $\epsilon(i) = \pm 1$. Show the p-fold wedge products $e^{i_1}\wedge\cdots\wedge e^{i_p}$ form an orthonormal basis of p-forms with $\langle e^{i_1}\wedge\cdots\wedge e^{i_p}, e^{i_1}\wedge\cdots\wedge e^{i_p}\rangle = \epsilon(i_1)\cdots\epsilon(i_p)$.*

From exercise 52 we already know that the nondegeneracy of g is equivalent to the invertibility of the matrix of g applied to basis elements. The same will be true here. First recall that the terms in the p-fold wedge products must be distinct so that the p-form doesn't vanish. Suppose we have two basis p-forms in which m basis 1-forms appear in both. Reordering so that these come first, the matrix of $g(e^{i_j}, e^{k_\ell})$ will be block diagonal with the first $m\times m$ block equal to the identity matrix, and zero everywhere else. This matrix obviously has zero determinant, thus the basis p-forms with any distinct elements are orthogonal. Otherwise, the determinant yields the product of the individual normalizations, establishing the last equation posed in the problem. This in turn means that the metric matrix for the basis p-forms is diagonal with entries ± 1, whence it has full rank and is therefore nondegenerate.

58 *Let $E = E_j dx^j$ be a 1-form on $\mathbb{R}^3$ with its Euclidean metric. Show that $\langle E, E\rangle = E_x^2 + E_y^2 + E_z^2$. Similarly, let $B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ be a 2-form. Show that $\langle B, B\rangle = B_x^2 + B_y^2 + B_z^2$.*

$\langle E, E\rangle = \delta^{i,j} E_i E_j = E_x^2 + E_y^2 + E_z^2$. $\langle B, B\rangle = \langle B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy\rangle = B_x^2 + B_y^2 + B_z^2$ since the different wedge products are orthonormal.

59 *In $\mathbb{R}^4$ let F be the 2-form given by $F = B + E \wedge dt$, where E and B are given by the formulas in the previous exercise. Using the Minkowski metric, calculate $-\frac{1}{2}\langle F, F\rangle$.*

$\langle F, F\rangle = \langle B + E \wedge dt, B + E \wedge dt\rangle = \langle B, B\rangle + \langle B, E \wedge dt\rangle + \langle E \wedge dt, B\rangle + \langle E \wedge dt, E \wedge dt\rangle$. The middle two terms are zero by orthogonality of different wedge products, leaving only the last term to deal with. $\langle E \wedge dt, E \wedge dt\rangle = \det \begin{pmatrix} \langle E, E\rangle & \langle E, dt\rangle \\ \langle dt, E\rangle & \langle dt, dt\rangle \end{pmatrix} = \det \begin{pmatrix} \langle E, E\rangle & 0 \\ 0 & -1 \end{pmatrix} = -\langle E, E\rangle$. Hence $-\frac{1}{2}\langle F, F\rangle = \frac{1}{2}(\langle E, E\rangle - \langle B, B\rangle)$, which is the Lagrangian of the source(charge)-free field.

60 *Show that any even permutation of a given basis has the same orientation, while any odd permutation has the opposite orientation.*

To check this we must examine the determinant. Since the determinant of a product is the product of determinants, we can break up the permutation of basis elements into a product of pairwise swaps, whose individual determinant is minus one. Thus, if the permutation has an even number of swaps, the new basis has the same orientation, while an odd number of swaps implies the determinant will equal minus one.

61 *Let M be an oriented manifold. Show that we can cover M with oriented charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, that is, charts such that the basis dx^μ of cotangent vectors on $\mathbb{R}^n$, pulled back to U^α by ϕ_α, is positively oriented.*

The point here is that the volume form on $\mathbb{R}^n$ can be pulled back to U_α by ϕ_α. Since M is orientable, we can define a volume form ω on all of M to specify the standard orientation. In the chart U_α we can also define a volume form by pulling back the standard volume form $\bigwedge_\mu dx^\mu$ from $\mathbb{R}^n$. If this specifies a different orientation, just reorder the basis of $\mathbb{R}^n$ so as to have the same orientation as given by ω. Since ω exists in all charts U_α, we can always orient these appropriately.

62 *Given a diffeomorphism $\phi : M \to N$ from one oriented manifold to another, we say that ϕ is orientation-preserving if the pullback of any right-handed basis (standard orientation) of a cotangent space in N is a right-handed basis of a cotangent space in M. Show that if we can cover M with charts such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are orientation-preserving, we can make M into an oriented manifold by using the charts to transfer the standard orientation on $\mathbb{R}^n$ to an orientation on M.*

In each chart pull the standard volume form on $\mathbb{R}^n$ back to M. Since any two charts are connected by an orientation-preserving transition function, the orientation is the same in any chart and the manifold is oriented everywhere.

63 *Let M be an oriented n-dimensional semi-Riemannian manifold and let $\{e_\mu\}$ be an oriented orthonormal basis of cotangent vectors at some point $p \in M$. Show that $e_1 \wedge \cdots \wedge e_n = \mathrm{vol}_p$ where* vol *is the volume form associated to the metric on M, and* vol_p *is its value at p.*

The oriented basis in question can be generated from the standard basis by applying a transformation T. Since the input and output are orthonormal bases, T is an orthogonal matrix, implying $\det T = \pm 1$. Because it's orientation preserving, $\det T = 1$. Since in transforming $e_1 \wedge \cdots \wedge e_n$ into the standard basis at p, we pick up a factor equal to the determinant of T, $e_1 \wedge \cdots \wedge e_n = \mathrm{vol}_p$.

64 *Show that if we define the Hodge star operator in a chart using the formula $\star(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \pm e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}$, where the second set of indices consists of those not contained in the first and the $\pm$ is given by $\mathrm{sign}(i_1, \ldots, i_n)\epsilon(i_1)\cdots\epsilon(i_p)$, it satisfies the property $\omega \wedge \star\mu = \langle \omega, \mu \rangle \mathrm{vol}$.*

First expand ω and μ: $\omega = \omega_\alpha e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_p}$, $\mu = \mu_\beta e^{\beta_1} \wedge \cdots \wedge e^{\beta_p}$. Now compute their inner product: $\langle \omega, \mu \rangle = \omega_\alpha \mu_\beta \langle e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_p}, e^{\beta_1} \wedge \cdots \wedge e^{\beta_p} \rangle = \omega_\alpha \mu_\beta \delta^{\alpha,\beta} \epsilon(\alpha_1) \cdots \epsilon(\alpha_p)$. Now $\star\mu = \pm \mu_\alpha e^{\alpha_{p+1}} \wedge \cdots \wedge e^{\alpha_n}$, and taking the wedge product with ω, one obtains $\omega \wedge \star\mu = \pm \omega_\alpha \mu_\beta \delta^{\alpha,\beta} e^{\alpha_1} \wedge \cdots e^{\alpha_n}$. Finally, $e^{\alpha_1} \wedge \cdots e^{\alpha_n} = \mathrm{sign}(\alpha_1, \ldots, \alpha_n)\mathrm{vol}$, so we obtain the desired result.

65 *Calculate $\star d\omega$ when ω is a 1-form on $\mathbb{R}^3$.*

$\star d\omega$ is essentially the curl of ω. First expand ω in an orthonormal basis e^j: $\omega = \omega_j e^j$. $d\omega = \partial_k \omega_j dx^k \wedge dx^j$. Finally, $\star d\omega = \partial_j \omega_k \, \mathrm{sign}(j, k, \ell) dx^\ell = \partial_j \omega_k \epsilon^{jk}_{\ell} dx^\ell$. Note that $\epsilon(i) = 1$ for all i in this case.

66 *Calculate $\star d \star \omega$ when ω is a 1-form on $\mathbb{R}^3$.*

This gives the divergence of ω. Expanding ω in a basis as above, we obtain $\star\omega = \omega_j \epsilon^j_{k\ell} dx^k \wedge dx^\ell$. Applying d gives $d \star \omega = \partial_m \omega_j \epsilon^j_{k\ell} dx^m \wedge dx^k \wedge dx^\ell$. Another $\star$ yields the final result: $\star d \star \omega = \partial_m \omega_j \epsilon^j_{k\ell} \epsilon^{mk\ell}$ and since $\epsilon^j_{k\ell}\epsilon^{mk\ell} = \delta^{jm}$, we have $\star d \star \omega = \partial_j \omega_k \delta^{jk}$, the divergence.

67 *Give $\mathbb{R}^4$ the Minkowski metric and the orientation in which (dt, dx, dy, dz) is positively oriented. Calculate the Holdge star operator on all wedge products of dx^μ's Show that on p-forms $\star^2 = (-1)^{p(4-p)+1}$.*

First we have the definition vol$=$ $dt \wedge dx \wedge dy \wedge dz$. Using this we can make a table of the various starred wedge products.

ω	$\star\omega$
dt	$-dx \wedge dy \wedge dz$
dx	$-dt \wedge dy \wedge dz$
dy	$dt \wedge dx \wedge dz$
dz	$-dt \wedge dx \wedge dy$
$dt \wedge dx$	$-dy \wedge dz$
$dt \wedge dy$	$dx \wedge dz$
$dt \wedge dz$	$-dx \wedge dy$
$dx \wedge dy$	$dt \wedge dz$
$dx \wedge dz$	$-dt \wedge dy$
$dy \wedge dz$	$dt \wedge dx$
$dt \wedge dx \wedge dy$	$-dz$
$dt \wedge dx \wedge dz$	dy
$dt \wedge dy \wedge dz$	$-dx$
$dx \wedge dy \wedge dz$	$-dt$

From the table we immediately see that the given condition is fulfilled.

68 *Let M be an oriented semi-Riemannian manifold of dimension n and signature $(s, n-s)$. Show that on p-forms $\star^2 = (-1)^{p(n-p)+s}$.*

$\omega\wedge\star\omega = \langle\omega,\omega\rangle\text{vol}$ and $\star\omega\wedge\star^2\omega = \langle\star\omega,\star\omega\rangle\text{vol}$, so $\star\omega\wedge\star^2\omega = (-1)^{p(n-p)}\star^2\omega\wedge\star\omega = \frac{\langle\star\omega,\star\omega\rangle}{\langle\omega,\omega\rangle}\,\omega\wedge\star\omega$. Hence $\star^2 = (-1)^{p(n-p)}\frac{\langle\star\omega,\star\omega\rangle}{\langle\omega,\omega\rangle}$. To evaluate this expression, let ω be a basis p-form $e^{i_1}\wedge\cdots\wedge e^{i_p}$. Setting $\epsilon(\mu) = \langle e^\mu, e^\mu\rangle$, we have $\langle\omega,\omega\rangle = \prod_{j=1}^p \epsilon(i_j)$. On the other hand, $\star\omega = \pm e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$, so $\langle\star\omega,\star\omega\rangle = \prod_{j=p+1}^n \epsilon(i_{p+j})$. Since all the terms in the denominator are ±1, we might as well multiply by $\langle\omega,\omega\rangle$ instead of dividing. We then obtain $\star^2 = (-1)^{p(n-p)}\prod_{j=1}^n \epsilon(i_j)$. The latter term equals $(-1)^s$ as s is the number of basis elements with norm -1.

69 *Let M be an oriented semi-Riemannian manifold of dimension n and signature $(s, n-s)$. Let e^μ be an orthonormal basis of 1-forms on some chart. Define the Levi-Civita symbol for $1 \le i_j \le n$ by*

$$\epsilon_{i_1\dots i_n} = \begin{cases} \text{sign}(i_1\dots i_n) & \textit{all } i_j \textit{ distinct} \\ 0 & \text{otherwise} \end{cases}$$

Show that for any p-form $\omega = \frac{1}{p!}\omega_{i_1\dots i_p} e^{i_1}\wedge\cdots\wedge e^{i_p}$ we have $(\star\omega)_{j_1\dots j_{n-p}} = \frac{1}{p!}\epsilon^{i_1\dots i_p}_{j_1\dots j_{n-p}}\omega_{i_1\dots i_p}$.

Using the results of exercise 64, we have $\star(e^{i_1}\wedge\cdots\wedge e^{i_p}) = \epsilon(i_1)\cdots\epsilon(i_p)\epsilon^{i_1\dots i_p}_{i_{p+1}\dots i_n} e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$ (no sum). Now $(\star\omega)_{j_1\dots j_{n-p}} = \epsilon(j_1)\cdots\epsilon(j_{n-p})\langle e^{j_1}\wedge\cdots\wedge e^{j_{n-p}},\star\omega\rangle$, so $(\star\omega)_{j_1\dots j_{n-p}} = \frac{(-1)^s}{p!}\epsilon^{i_1\dots i_p}_{j_1\dots j_{n-p}}\omega_{i_1\dots i_p}$.

70 *Show that the Maxwell equations $\nabla\cdot\vec{E} = \rho$, $\nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$ can be rewritten as $\star_S d_S \star_S E = \rho$ and $-\partial_t E + \star_S d_S \star_S B = j$.*

Start from $E = E_j dx^j$. For the remainder of the exercise, $\star$ means $\star_S$. So $\star E = \frac{1}{2}E_j\epsilon^j_{k\ell}dx^k\wedge dx^\ell$. Then $d\star E = \frac{1}{2}\epsilon^j_{k\ell}\partial_m E_j dx^m\wedge dx^k\wedge dx^\ell$, and finally $\star d\star E = \frac{1}{2}\epsilon^j_{k\ell}\epsilon^{mk\ell}\partial_m E_j = \partial_j E_j = \nabla\cdot\vec{E}$. Meanwhile $B = \frac{1}{2}\epsilon^j_{k\ell}B_j dx^k\wedge dx^\ell$. Thus $\star B = \frac{1}{2}\epsilon^j_{k\ell}\epsilon^{k\ell}_{m}B_j dx^m = B_j dx^j$. Then $d\star B = \partial_k B_j dx^k\wedge dx^j$ and $\star d\star B = \epsilon_\ell^{kj}\partial_k B_j dx^\ell$, the components of which are simply $\nabla\times\vec{B}$.

71 *Show that on a semi-Riemannian manifold M which can be decomposed into $\mathbb{R}\times S$, where S is space, the general Maxwell equations $dF = 0$ and $\star d\star F = J$ can be transformed into their* *"usual" appearance $d_SE = 0, \partial_t B + d_SE = 0, \star_S d_S \star_S E = \rho,$ and $\star_S d_S \star_S B - \partial_t E = j$.*

Given the decomposition of M, we write $F = B + E \wedge dt$ where B is the portion of F defined only on S. Similarly $J = j - \rho dt$. The first equation then reads $dB + dE \wedge dt = 0$. Observe that $dB = d_SB + \partial_t B \wedge dt$ while $dE \wedge dt = \partial_t E dt \wedge dt + d_SE \wedge dt$. Thus we have $d_SB = 0$ and $\partial_t B + d_SE = 0$; the first equation deals with three forms only on space, while the second involves three forms on space and time. Assume further that the metric can be decomposed into $g = -dt^2 + {}^3g$ (can this always be done when the underlying space is a product? The tangent space can be decomposed, it seems clear, from the picture of vectors as arrows pointing tangentially to the surface. In the case of a product space, there are arrows for each manifold, and we take their formal product to get the tangent to the total manifold. But from the point of view of vectors as derivatives? I think it might be possible by showing that the value of the derivative is equal to making changes on each submanifold separately and then adding them. In other words, we have a curve $\gamma(t)$ on M, but since $M = \mathbb{R} \times S$, we can write this as $(\gamma_1(t), \gamma_2(t))$. Now $\gamma'(t)[f] = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_1(t), \gamma_2(t)) = \frac{\partial f}{\partial \gamma_1}\frac{\partial \gamma_1}{\partial t} + \frac{\partial f}{\partial \gamma_2}\frac{\partial \gamma_2}{\partial t} = \gamma_1'(t)[f]|_{\gamma_2(t)} + \gamma_2'[f]|_{\gamma_1(t)} \equiv (\gamma_1'(t), \gamma_2'(t))[f]$. That's what we do in $\mathbb{R}^n$, after all. I guess it's obvious once you map open sets of the manifold to $\mathbb{R}^n$. But this doesn't mean the metric has to be block diagonal and respect the split; after all that *doesn't* always happen in $\mathbb{R}^n$.) Now let $\star_S$ be the Hodge dual on (differential forms on) S. Since E is a one form on S, $\star E \wedge dt = \star_S E$ and similarly since B is a two form on S, $\star B = -\star_S B \wedge dt$ (using the chart from 67). So $\star F = \star_S E - \star_S B \wedge dt$ and then $d \star F = d_S \star_S E + \star_S(\partial_t E) \wedge dt - d_S \star_S B \wedge dt$. Note that the first term is a three form on space, so $\star d_S \star_S E$ must be a one form on time, $\star d_S \star_S E = -(\star_S d_S \star_S E)dt$, where the minus sign can be determined again by the chart from 67. The second term is a three form on space and time, and due to the ordering we'll have $\star(\star_S(\partial_t E) \wedge dt) = -\partial_t E$. The last term is also a three form on space and time and we end up with $\star(d_S \star_S B \wedge dt) = -\star_S d_S \star_S B$. Thus $\star_S d_S \star_S E = \rho$ and $-\partial_t E + \star_S d_S \star_S B = j$.

72 *Show that in a Riemannian 4-dimensional manifold any 2-form F can be written as a sum of self-dual and anti-self-dual parts, $F = F_+ + F_-$ for $\star F_\pm = \pm F_\pm$, if we take $F_\pm = (F \pm \star F)/2$.*

Clearly $F = F_+ + F_-$. And $\star F_\pm = (\star F \pm \star^2 F)/2 = (\pm F + \star F)/2 = \pm(F \pm \star F)/2 = \pm F_\pm$.

73 *Show that in a Lorentizan 4-manifold any 2-form F can be written as a sum of self-dual and anti-self-dual parts, $F = F_+ + F_-$ for $\star F_\pm = \pm i F_\pm$.*

Take $F_\pm = (F \mp \star i F)/2$. Then $\star F_\pm = (\star F \mp \star^2 i F)/2 = (\pm i F + \star F)/2 = \pm i(F \mp \star i F)/2 = \pm i F_\pm$.

74 *Show that the equations $\star_S E = iB$ and $\star_S B = -iE$ are equivalent and that they both hold if at every time t we have $E = E_i dx^i$ and $B = -(i/2)\varepsilon^j{}_{k\ell} E_j dx^k \wedge dx^\ell$.*

Applying $\star_S$ turns one equation into the other, to they are equivalent. From exercise 69 we know that $\star_S E = (1/2)\varepsilon^j{}_{k\ell} E_j dx^k \wedge dx^\ell$ which is clearly equal to iB.

75 *Show that the second Maxwell equation $\partial_t B + d_S E = 0$ leads to ${}^3k \wedge E = k_0 B$.*

Start with $d_S E = d_S(e^{ik_\mu x^\mu} E_j dx^j) = \partial_k(e^{ik_\mu x^\mu} E_j) dx^j \wedge dx^k = i k_k E_j dx^j \wedge dx^k = -i {}^3k \wedge E$. Meanwhile $\partial_t B = i k_0 B$, so $\partial_t B + d_S E = i(k_0 B - {}^3k \wedge E) = 0$, the desired result.

76 *Show that ${}^3k \wedge E = -ik_0 \star_s E$ implies $k_\mu k^\mu = 0$.*

Start from $\langle {}^3k \wedge E, {}^3k \wedge E \rangle = k_0^2 \langle \star_s E, \star_s E \rangle$. Note that the inner product should be antilinear in one of its inputs. Using the coordinate expressions for E and ${}^3k$ we have $\langle {}^3k \wedge E, {}^3k \wedge E\rangle = k_\ell k^*_{\ell'} E_j E^*_{j'} \langle dx^\ell \wedge dx^j, dx^{\ell'} \wedge dx^{j'}\rangle = k_\ell k^*_{\ell'} E_j E^*_{j'} (\delta^{jj'}\delta^{\ell\ell'} - \delta^{j'\ell}\delta^{j\ell'}) = \langle {}^3k, {}^3k\rangle \langle E, E\rangle - |\langle {}^3k, E\rangle|^2 = \langle {}^3k, {}^3k\rangle \langle E, E\rangle$, since E is orthogonal to ${}^3k$. On the other hand, $k_0^2 \langle \star_s E, \star_s E\rangle = \frac{k_0^2}{4} \varepsilon^j_{k\ell} \varepsilon^{j'}_{k'\ell'} E_j E_{j'} \langle dx^k \wedge dx^\ell, dx^{k'} \wedge dx^{\ell'}\rangle = \frac{k_0^2}{4} \varepsilon^j_{k\ell} \varepsilon^{j'}_{k'\ell'} E_j E_{j'} (\delta^{kk'}\delta^{\ell\ell'} - \delta^{k\ell'}\delta^{k'\ell}) = k_0^2 \langle E, E\rangle$. Thus, $\langle {}^3k, {}^3k\rangle = k_0^2$. Using the metric, we have $k_0^2 = -k_0 k^0$, and the other components are unchanged, so $k_\mu k^\mu = 0$.

77 *Show $\vec{E} = (0, e^{i(t-x)}, -ie^{i(t-x)})$, $\vec{B} = i\vec{E}$ satisfy the vacuum Maxwell equations. [corrected from the book]*

Since $\vec{\nabla} \cdot \vec{E} = 0$ (by inspection), $\vec{\nabla} \cdot \vec{B} = 0$, too. Now $\vec{\nabla} \times \vec{E} = \vec{E}$ and $\partial_t \vec{B} = i\partial_t \vec{E} = -\vec{E}$, so $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$. Finally, $\vec{\nabla} \times \vec{B} = i\vec{E}$, so $\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 0$.

78 *Prove that all self-dual and anti-self-dual plane wave solutions are left and right circularly polarized, respectively.*

Without loss of generality we can take ${}^3k$ to point in the x direction. Then the self-dual case is worked out in detail in the book. Left circular polarization can be recognized from the form of E, namely that the polarization vector is proportional to $(1, -i)$; right circular polarization is given by $(1, i)$. In the anti-self-dual case $\star_S E = -iB$ or, equivalently, $\star_S B = iE$. The first Maxwell equation $d_S B = 0$ reads as before, $B \wedge {}^3k = 0$, so $\langle E, {}^3k\rangle = 0$ still holds. The second equation, $\partial_t B + d_S E = 0$, leads to ${}^3k \wedge E = k_0 B$ as before. Rewriting this in terms of E we obtain ${}^3k \wedge E = ik_0 \star_S E$, which again leads to $\langle {}^3k \wedge E, {}^3k \wedge E\rangle = k_0^2 \langle \star_s E, \star_s E\rangle$, as in exercise 76. Thus we know that k must be lightlike, so we can assume without loss of generality that $k = dt - dx$. E must be orthogonal to ${}^3k$, so $E = ady + bdz$. We now use the second Maxwell equation to determine a and b. ${}^3k \wedge E = -(adx \wedge dy + bdx \wedge dz)$, while $ik_0 \star_S E = i(adz \wedge dx + bdx \wedge dy)$, so $b = ia$, which is the condition for right circular polarization.

79 *Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ be the parity transformation $P(t, x, y, z) = (t, -x, -y, -z)$. Show that if F is a self-dual solution of Maxwell's equations, the pullback P^*F is an anti-self-dual solution, and vice versa.*

We don't really need to show both, since the vice versa part follows from $P^*P^*F = F$. To see that P^*F takes a self-dual solution to an anti-self-dual solution, note that $P^*E = -E$, while $P^*B = B$. (This is why B is called an axial vector sometimes.) If $F = dE \wedge dt + B$ was self-dual to begin with, meaning $\star_S E = iB$, the new $F' = dE' \wedge dt' + B' = -dE \wedge dt + B$ is anti-self-dual: $\star_S E' = - \star_S E = -iB = -iB'$.

Chapter 6

80 *Show that $\omega = (xdy - ydx)/(x^2 + y^2)$ is closed. Show that $\int_{\gamma_0} \omega = -\pi$ while $\int_{\gamma_1} \omega = \pi$.*

Let $r = x^2 + y^2$. Then $\partial_i r = 2x_i$. Now $d\omega = \partial_x(x/r) dy \wedge dx - \partial_y(y/r) dx \wedge dy = (1/r - 2x^2/r^2) dy \wedge dx + (1/r - 2y^2/r^2) dy \wedge dx = (y^2 - x^2)/r^2 dy \wedge dx + (x^2 - y^2)/r^2 dy \wedge dx = 0$. Turning to the integrals, γ_0 is the upper half of the unit circle, traversed from (-1,0) to (1,0). We can parameterize this path as $\gamma_0(t) = (\cos \pi(1-t), \sin \pi(1-t))$, and thus $\gamma_0'(t) = \pi(\sin \pi(1-t), -\cos \pi(1-t))$. In these coordinates and on this path $\omega = (-y/r, x/r) = (-\sin \pi(1-t), \cos \pi(1-t))$, so $\int_{\gamma_0} \omega = \int_0^1 \omega_{\gamma_0(t)}(\gamma_0'(t)) \mathrm{d}t = -\pi$. The other curve is $\gamma_1(t) = (\cos \pi(1-t), -\sin \pi(1-t))$, so $\gamma_1'(t) = \pi(\sin \pi(1-t), \cos \pi(1-t))$ and $\omega_{\gamma_1(t)} = (\sin \pi(1-t), \cos \pi(1-t))$, leading to $\int_{\gamma_1} \omega = \pi$.

81 *Show that $\mathbb{R}^n$ is simply connected by exhibiting an explicit formula for a homotopy between any two paths between arbitrary points $p, q \in \mathbb{R}^n$.*

Let $f(t)$ and $g(t)$ be paths from p to q. Then define $h(s,t) = sf(t) + (1-s)g(t)$. Clearly $h(0,t) = f(t)$ and $h(1,t) = g(t)$ and since all the intervening points are elements of $\mathbb{R}^n$ for all t, h is a homotopy between f and g.

82 *Show that a 1-form E is exact iff $\int_\gamma E = 0$ for all loops γ. (Hint: if E is not exact, show that there are two smooth paths γ, γ' from some point $x \in M$ to some point $y \in M$ such that $\int_\gamma \omega \neq \int_{\gamma'} E$. Use these paths to form a loop, perhaps only piecewise smooth.*

Start with the 'if' statement and let $E = d\phi$ be an exact form and γ a loop starting and ending at p. Then $\oint_\gamma E = \int_0^1 d\phi(\gamma'(t))\mathrm{d}t = \int_0^1 \gamma'(t)[\phi]\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma(s))_{s=t}\mathrm{d}t = \int_0^1 [\phi(\gamma(t))]'\mathrm{d}t = \phi(\gamma(1)) - \phi(\gamma(0)) = \phi(p) - \phi(p) = 0$. Now the 'only if' part, which we will show by proving the contrapositive. If E is closed but not exact, then the manifold must not be simply connected. Then choose two points x and y such that the paths γ, γ' have no homotopy between them, i.e. γ and γ' go on either side of the "hole" from x to y; together they encircle the hole. Additionally, we must be able to find some points x, y and paths γ, γ' such that $\int_\gamma E \neq \int_{\gamma'} E$ or else we could use this integral to define ϕ for which $E = d\phi$. Then defining $\tilde{\gamma}$ to be the path taken by following γ forwards and then γ' backwards, we have found a path $\tilde{\gamma}$ such that $\oint_{\tilde{\gamma}} E \neq 0$.

83 *For any manifold M, show the manifold $S^1 \times M$ is not simply connected by finding a 1-form on it that is closed by not exact.*

Picking coordinates (t, x^j) consider the 1-form ω =(1,0,. . . ,0). Clearly $d\omega = 0$ so the form is closed, but it is not exact. There is no continuous function ϕ defined on the whole manifold such that $\omega = d\phi$, since if we try to define it using $\int_\gamma \omega$ where γ is a path around S^1 and fixed on M, we obtain $\phi(t, x^j) = t$, which is multiple-valued.

84 *Let the n-disk D^n be defined as $D^n = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 \leq 1\}$. Show that D^n is an n-manifold with boundary in an obvious sort of way.*

Locally the neighborhood of each point on the unit circle S^n already looks a lot like H^n where the half space is defined by the tangent hyperplane. In particular, we can make suitable charts by first using spherical coordinates to describe the points in the disk and then for each point, assuming that it lies at the north pole and expanding the coordinates in a Taylor series in θ to first order. In two dimensions we then obtain $r(\cos\theta, \sin\theta) \to r(1, \theta)$ and in three dimensions $r(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \to r(1, \theta\cos\varphi, \theta\sin\varphi)$. We then obtain a cylindrical coordinatization of the neighborhood of the selected point.

85 *Check that the definition of tangent vectors in Chapter 3 really does imply that the tangent space at any point on the boudary of an n-dimensional manifold with boundary is an n-dimensional vector space.*

Tangent vectors are simply maps from C^∞ to $\mathbb{R}$ obeying linearity and the Leibniz rule. If we first map the point on the boundary to the boundary of H^n, where $x^n \geq 0$ is the special coordinate, then obviously we still have all the derivatives in the directions $x_1, \ldots, x_{n-1}$. The derivative in the x_n direction still works as well, since the functions must be smooth to a small region outside the boundary, for which $-\varepsilon < x_n < 0$.

86 *For the mathematically-inclined reader: prove that $\int_M \omega$ is independent of the choice of charts and partition of unity.*

87 *Show that $\partial D^n = S^{n-1}$.*

In exercise 84 we showed a collection of charts in which at every point on the surface of the sphere nearby surface points are mapped to points on ∂H^n, thought of as the tangent hyperplane. Thus, surface points are on the boundary. No other points can be on the boundary since they are "surrounded" by other points in the manifold. Consider mapping a point in the interior to a point on ∂H^n. Not all open sets containing the point are mapped to open sets of H^n, and thus the chart transition function isn't smooth (isn't even continuous).

88 *Let $M = [0,1]$. Show that Stokes' theorem in this case is equivalent to the fundamental theorem of calculus $\int_0^1 \mathrm{d}x\, f'(x) = f(1) - f(0)$.*

Consider the 1-form $\omega = df$. By Stokes' theorem $\int_M \omega = \int_0^1 df = \int_{\partial M} f$. The boundary ∂M is manifestly equal to $\{0,1\}$; only the orientation remains to be determined. dx defines the orientation by defining increasing x to be outward-facing at 1 and inward-facing at 0. Thus $\int_{\partial M} f = f(1) - f(0)$.

89 *Let $M = [0,\infty)$, which is not compact. Show that without the assumption that f vanishes outside a compact set, Stokes' theorem may not apply. (Hint: in this case Stokes' theorem says $\int_0^\infty \mathrm{d}x\, f'(x) = -f(0)$.)*

Choose the function $f'(x) = 1$. Then $f(x) = x$. The integral clearly diverges, whereas by Stokes' theorem it would be zero. The function $f'(x) = e^{-x}$ works, though.

90 *Show that any submanifold is a manifold in its own right in a natural way.*

A manifold is defined by having charts φ_α which smoothly map the open sets U_α to $\mathbb{R}^k$, where smooth means that the transition function $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth (infinitely differentiable) where defined. For a submanifold $S \subset M$ we can define the open sets to be of the form $V_\alpha = S \cap U_\alpha$; this is the induced topology. Charts φ_α of M are guaranteed to satisfy $S \cap U_\alpha = \varphi_\alpha^{-1}\mathbb{R}^k$, which means that every open set in S_α is the preimage of some hyperplane $\simeq \mathbb{R}^k$ under the chart φ_α. So we can just define ϑ_α to be this map from S_α to $\mathbb{R}^k$. The transition functions $\vartheta_\alpha \circ \vartheta_\beta^{-1}$ are smooth because they are equivalent to the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$, restricted to the corresponding input and output hyperplanes.

91 *Show that S^{n-1} is a compact submanifold of $\mathbb{R}^n$.*

Use the maps in exercise 84, fixing $r = 1$.

92 *Show that any open subset of a manifold is a submanifold.*

Isn't this just exercise 4?

93 *Show that if S is a k-dimensional submanifold with boundary of M, then S is a manifold with boundary in a natural way. Moreover, show that ∂S is a $(k-1)$-dimensional submanifold of M.*

Same as exercise 90 except that the hyperplanes have boundaries. That doesn't change the fact that the transition functions will be smooth. ∂S is also a manifold using the charts from S with the modification that the image of the chart is now just the boundary of the hyperplane used before. Again, the transition functions will be smooth, since they are just restrictions of smooth maps.

94 *Show that D^n is a submanifold of $\mathbb{R}^n$ in this sense.*

Isn't this just exercise 84?

95 *Suppose that $S \subset \mathbb{R}^2$ is a 2-dimensional compact orientable submanifold with boundary. Work out what Stokes' theorem says when aplied to a 1-form on S. This is sometimes called Green's theorem.*

Let $\omega = \alpha dx + \beta dy$. Then $d\omega = (\partial_x\alpha - \partial_y\beta)dx \wedge dy$. Thus $\int_{\partial M} \alpha dx + \beta dy = \int_M (\partial_x\alpha - \partial_y\beta)dx \wedge dy$. To evaluate the lefthand side one should make a coordinate transformation by regarding ∂M as a curve $\gamma(t)$, the inverse of which we can regard as a map from ∂M to $\mathbb{R}$. This allows us to express dx and dy in terms of the coordinate 1-form dt: $dx = \frac{d\gamma^1}{dt}dt$ and likewise for dy.

96 *Suppose that $S \subset \mathbb{R}^3$ is a 2-dimensional compact orientable submanifold with boundary. Show that Stokes' theorem aplied to S boils down to the classic Stokes's theorem.*

Suppose we work in a patch of S with coordinates x, y, z such that z is fixed, so that $z = c$ defines S. On this patch let $\omega = v_i dx^i$, where $i = 1, 2$. Then $d\omega = \partial_j v_i dx^j \wedge dx^i$ for $i, j = 1, 2$. This can be written $d\omega = (\partial_2 v_1 - \partial_1 v_2)dx^1 \wedge dx^2$; the corresponding integral $\int_S d\omega = \int_S (\nabla \times \vec{v}) \cdot \hat{z} dA$, where A is the area (volume) element of S. The boundary of S must be specified by a curve $\gamma(t) = (x^1(t), x^2(t))$. It then follows that $\omega = v_i \frac{\partial x^i}{\partial t} dt$ so that the integral can be written $\int_{\partial S} \omega = \int \vec{v} \cdot d\vec{\ell}$ with $\vec{\ell} = \frac{\partial x^i}{\partial t}\hat{x}_i$. Thus we have $\int_S (\nabla \times \vec{v}) \cdot d\vec{A} = \int \vec{v} \cdot d\vec{\ell}$, the usual Stokes' theorem.

97 *Suppose that $S \subset \mathbb{R}^3$ is a 3-dimensional compact orientable submanifold with boundary. Show that Stokes's theorem aplied to S is equivalent to Gauss's theorem, also known as the divergence theorem.*

Let $\omega = \alpha dx \wedge dy + \beta dz \wedge dx + \gamma dy \wedge dz$. Then $d\omega = \partial_z \alpha dz \wedge dx \wedge dy + \partial_y \beta dy \wedge dz \wedge dx + \partial_x \gamma dx \wedge dy \wedge dz = (\partial_z\alpha + \partial_y\beta + \partial_x\gamma)dx \wedge dy \wedge dz$. Now $\int_{\partial M} \omega = \int_M d\omega$, the righthand side of which is simply $\int_S (\partial_z\alpha + \partial_y\beta + \partial_x\gamma)dxdydz$. If we define $v \equiv \star\omega = \alpha dz + \beta dy + \gamma dx$, then this integral is just $\int \partial_i v^i dxdydz$, i.e. the integral of the gradient of the vector $\vec{v}$. Meanwhile, consider a patch of ∂S covered by coordinates (s, t); that is, a specification of the points on ∂S: $x(s, t), y(s, t)$, and $z(s, t)$. Letting $dA = ds \wedge dt$ be the volume element on ∂S, then $dx \wedge dy = (\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s})dA$, and similarly for the other 2-forms. Then the lefthand integrand can be written as $\vec{v} \cdot d\vec{A}$ where the vector $\vec{A} = \epsilon_{ijk}\hat{x}_i(\frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial t})$.

98 *Show that the pullback of a closed form is closed and the pullback of an exact form is exact.*

The pullback and exterior derivative commute, so if ω is closed, $d\omega = 0$ and $\phi^*\omega$ is also closed, since $d\phi^*\omega = \phi^*(d\omega) = 0$. Similarly, if ω is exact, then so is $\phi^*\omega$: $\phi^*\omega = \phi^*(d\alpha) = d(\phi^*\alpha)$.

99 *Show that given any map $\phi : M \to M'$ there is a linear map from $H^p(M')$ to $H^p(M)$ given by $[\omega] \mapsto [\phi^*\omega]$, where ω is any closed p-form on M'. Call this linear map $\phi^* : H^p(M') \to H^p(M)$. Show that if $\psi : M' \to M''$ is another map, then $(\psi\phi)^* = \phi^*\psi^*$.*

The linear map is just the pullback: since it preserves closed and exact forms, it preserves equivalence classes of closed forms. If ω, ω' are closed, so are $\phi^*\omega$ and $\phi^*\omega'$ and if $\omega - \omega' = d\mu$, then $\phi^*\omega - \phi^*\omega' = \phi^* d\mu = d\phi^*\mu$. And the pullback of the composition of maps is the reverse-ordered composition of pullbacks.

100 *Show that $\star j = f(r) r\, dr \wedge d\theta$ for $j = f(r)dz$ and $d\theta = \omega$ as in exercise 80.*

$\star j = f(r) \star dz = f(r) dx \wedge dy$. Now note that $r dr \wedge d\theta = (xdx + ydy)(xdy - ydx)/r^2 = dx \wedge dy$

since $rdr = xdx + ydy$.

101 *Show that* $\star d\theta = \frac{1}{r} dz \wedge dr$.

$\star d\theta = (x \star dy - y \star dx)/r^2 = (xdz \wedge dx - ydy \wedge dz)/r^2 = (dz \wedge (xdx + ydy))/r^2 = dz \wedge dr/r$.

102 *Check that* $d \star B = \star j$ *holds iff* $g'(r) = rf(r)$ *for* $B = \frac{g(r)}{r} dz \wedge dr$.

$d \star B = d(g(r)d\theta) = g'(r)dr \wedge d\theta = \star j = rf(r)dr \wedge d\theta$. Thus $g'(r) = rf(r)$ is necessary and sufficient.

103